RAN - 2 3 0 3 0 8 0 3 0 2 0 4 0 0 0 4

RAN-2303080302040004

M.Sc. (Sem. - II) Examination September - 2025

Mathematics (Paper - PGMTH : 2043) Integral Transforms - II

Time: 3 Hours] **[Total Marks: 70**

સૂચના : / Instructions

(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fill up strictly the above details on your answer book

(1) Answer all questions

(2) Figures to the right indicates marks

(3) Follow usual notations and conventions

Seat No.:

Student's Signature

Q.1 Attempt Any Two: **14**

(a) Define Hankel transform. If $H_n[f(x)] = \bar{f}_n(s)$, then prove that $H_n[f(ax)] = \frac{1}{a^2}\bar{f}_n\left(\frac{s}{a}\right)$. Find Hankel transform of $f(x) = \frac{e^{-ax}}{x}$ by taking kernel $xJ_0(sx)$.

(b) Prove that $H_n[f'(x)] = \bar{f}'_n(s) = \frac{s}{2n}[(n-1)\bar{f}_{n+1}(s) - (n+1)\bar{f}_{n-1}(s)]$.

(c) (i) Find inverse Hankel transform of $\bar{f}(s) = \frac{e^{-as}}{s^2}$ taking $n = 1$.

(ii) Find $H\left[\frac{1}{x}\right]$ taking $n = 0$. Also recover the original function by using inversion formula.

Q.2. Attempt Any Two: **14**

(a) Define Hilbert transform. Find Hilbert transform of rectangular pulse given by $f(t) = \begin{cases} 1; \ for\ |t| < a \\ 0; \ for\ |t| > a \end{cases}$

(b) Show that Hankel transform of order zero of the function $f(x)$ defined by $f(x) = \begin{cases} a^2 - x^2, 0 < x < a \\ \quad 0, x > a \end{cases}$ is given by $\frac{4a}{s^3}J_1(as) - \frac{2a^2}{s^2}J_0(as)$

(c) Find (i) $H^{-1}[s^{-2}e^{-as}, n = 1]$

(ii) $H^{-1}[e^{-as}, \quad n = 0]$

Q.3. Attempt Any Two: **14**

(a) Find finite Hankel transform of $\left[\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx}\right]$ of order n taking the kernel $xJ_n(s_i x)$, where s_i is the root of $J_n(s_i a) = 0$.

(b) Define Finite Hankel transform.

(i) Find finite Hankel transform of order zero for constant 'C'.

(ii) Find finite Hankel transform of x^n, $n > -1$ where $xJ_n(s_i x)$ is a kernel of the transform.

(c) Prove that finite Hankel transform of $f(x) = \frac{2^{1+n-m}}{\lfloor m-n}x^n(1-x^2)^{m-n-1}$ for $0 \le x < 1$ is $s^{n-m}J_m(s)$.

Q.4. Attempt Any Two: **14**

(a) Heat is applied at a constant rate Q per unit area per unit time to a circular area (disk) of radius 'a' in the plane $z = 0$ to an infinite solid of thermal conductivity 'k'. Show that steady temperature at a distance 'r' from the plane z = 0 is given by,

$\frac{Qa}{2k}\int_0^\infty e^{-sz} J_0(sr) J_1(sa)\, s^{-1} ds$ with the boundary conditions

$V \to 0$ as $r \to \infty$, $V \to 0$ as $|z| \to \infty$,

$$-k\frac{\partial V}{\partial z} = \frac{Q}{2},\ 0 \le r \le a,\ z = 0$$
$$= 0,\ r > a,\ z = 0$$

(b) Find potential $V(r, z)$ of the field due to a flat circular disk with its centre at origin and radius equal to 1. The axis of disk is along z-axis and it satisfies differential equation, $\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0,\ 0 \le r < \infty,\ z \ge 0$

subject to boundary conditions

$V = V_0$ when $z = 0,\ 0 \le r \le 1$,

$\frac{\partial V}{\partial t} = 0$ when $z = 0,\ r > 1$

(c) Use finite Hankel transform to solve differential equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} = \frac{1}{k}\frac{\partial V}{\partial t},\ 0 \le r < 1,\ t \ge 0,$$

where $\frac{\partial V}{\partial r} + hV = 0$, for $r = 1$ *and* $t > 0$

$V = 1$, for $t = 0$ *and* $0 \le r < 1$

Q.5. Attempt Any Two: **14**

(a) If $H_i[f(t)] = \overline{f_H}(x)$, then prove that

(i) $H_i[f(t + a)] = \overline{f_H}(x + a)$

(ii) $H_i[f(-at)] = -\overline{f_H}(-ax)$

(iii) $H_i[tf(t)] = x\,[\overline{f_H}(x)] + \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt$

(b) If $S[f(t)] = \bar{f}(z)$, then prove that

(i) $S[f'(t)] = -\frac{1}{z}f(0) - \frac{d}{dz}\bar{f}(z)$

(ii) $S[f''(t)] = -\left[\frac{1}{z}f'(0) + \frac{1}{z^2}f(0)\right] + \frac{d^2}{dz^2}\bar{f}(z)$

(c) Find S-transform of

(i) $f(t) = \frac{1}{t + a}$

(ii) $f(t) = \sin(k\sqrt{t})$